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LETTER TO THE EDITOR

A class of completely integrable hyperbolic systems

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Abstract. The subject of this letter is the characterisation of a class of quasilinear hyperbolic systems which enjoy the property of being separable in two independent equations. A sort of duality relating these with the Lax 'linearly degenerate' systems is also pointed out.

Let us consider a quasilinear hyperbolic system

$$M(U)U_t + N(U)U_x = 0 \tag{1}$$

where $U = {}^{\mathrm{T}}(U^1, U^2)$ is a column vector and M and N are two 2×2 matrices (with no explicit dependence on x and t). A system of the type (1) is called hyperbolic (Lax 1957) when $\det M \neq 0$ and the eigenvalue problem

$$(N - \lambda M)R = 0 \qquad \text{and} \qquad L(N - \lambda M) = 0$$

admits two distinct real eigenvalues $\lambda^k = \lambda^k(U)$ and two corresponding left $L^k = L^k(U)$ and right $R^k = R^k(U)$ eigenvectors. L^k and R^k are also supposed to satisfy the condition of normalisation

$$L^k M R^m = \delta^{km} \qquad (\forall m, k). \tag{2}$$

Left-multiplying (1) by L^k one gets (Jeffrey 1976)

$$L^k M(U_t + \lambda^k U_x) = 0 \qquad \text{with} \qquad \lambda^k = \lambda^k(U^1, U^2). \tag{3}$$

If the kth wave satisfies the condition

$$\varphi^k L^k M = \partial\lambda^k/\partial U \tag{4}$$

where $\varphi^k = \varphi^k(U)$ is an integrating factor, then (3) becomes

$$\lambda^k_t + \lambda^k \lambda^k_x = 0. \tag{5}$$

This is a very special case of a wave possessing a Riemann invariant α^k, which is characterised by the condition

$$\varphi^k L^k M = \partial\alpha^k/\partial U \tag{6}$$

and by the fact that α^k satisfies the equation

$$\alpha^k_t + \lambda^k \alpha^k_x = 0 \tag{7}$$

or the equivalent one

$$\frac{\mathrm{d}\alpha^k}{\mathrm{d}t} = 0 \qquad \text{along the curves} \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda^k$$

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for any solution U of (1) (cf Majda and Rosales 1984). We have evidently supposed that the Jacobian matrices $J(\partial\lambda^k/\partial U^m)$ and $J(\partial\alpha^k/\partial U^m)$ have non-vanishing determinants.

A physical example of a 2×2 hyperbolic system leading to the crucial condition (4) will be presented towards the end of the letter.

A comparison between (2) and (4) shows that the aforementioned special system (in which (4) holds for $k=1,2$) enjoys the property

$$\frac{\partial\lambda^k}{\partial U}R^m=0 \qquad \text{for } k\neq m \tag{8a}$$

and

$$\frac{\partial\lambda^k}{\partial U}R^k=\varphi^k\neq 0 \qquad \text{for } k=1,2. \tag{8b}$$

Clearly (7) reduces to (5) when

$$\alpha^k=\alpha^k(\lambda^k). \tag{9}$$

This fact means that equations (7) are decoupled from each other if each of the Riemann invariants is a function only of the corresponding (in our order) eigenvalue.

From now on we shall put $\lambda^1=A$, $\lambda^2=B$, $R^1=R^A$, $R^2=R^B$ and also $\alpha^1=\alpha$, $\alpha^2=\beta$. The conditions (8) and (9) then become

$$\frac{\partial A}{\partial U}R^B=\frac{\partial B}{\partial U}R^A=0 \qquad \frac{\partial A}{\partial U}R^A=\varphi^A \qquad \frac{\partial B}{\partial U}R^B=\varphi^B \tag{10}$$

and

$$\alpha=\alpha(A) \qquad \beta=\beta(B) \tag{11}$$

while (5) takes the form

$$A_t+AA_x=0 \qquad \text{and} \qquad B_t+BB_x=0. \tag{12}$$

A straightforward way of reducing system (7) (and in particular (12)) to a linear system is to carry out a 'hodograph transform', which consists in interchanging the roles of dependent and independent variables. So, provided that the Jacobian determinant of the transformation $(\alpha,\beta)\rightarrow(x,t)$ is non-zero, one has

$$\alpha_t=x_\beta/D \qquad \alpha_x=-t_\beta/D$$

$$\beta_t=-x_\alpha/D \qquad \beta_x=t_\alpha/D \qquad D=t_\alpha x_\beta-x_\alpha t_\beta$$

whereupon a simple insertion of the above expressions in (7) (i.e. $\alpha_t+A\alpha_x=0$ and $\beta_t+B\beta_x=0$) yields

$$x_\alpha-Bt_\alpha=0 \qquad x_\beta-At_\beta=0$$

where $x=x(\alpha,\beta)$, $t=t(\alpha,\beta)$, $A=A(\alpha,\beta)$ and $B=B(\alpha,\beta)$.

After equating the cross-derivatives of $x(\alpha,\beta)$ one gets the following Euler–Poisson–Darboux equation for $t(\alpha,\beta)$:

$$(A-B)t_{\alpha\beta}+A_\alpha t_\beta-B_\beta t_\alpha=0.$$

Analogously one can derive an equation for $x(\alpha,\beta)$.

The hodograph system associated with (12) is

$$x_A = Bt_A \qquad \text{and} \qquad x_B = At_B. \tag{13}$$

On elimination of $x(A, B)$ in (13) one gets

$$\xi_{AB} = 0 \qquad \text{with} \qquad \xi = (A - B)t \tag{14}$$

or, alternatively, by eliminating $t(A, B)$ one obtains from (13) that

$$\tau_{AB} = 0 \qquad \text{with} \qquad \tau = \left(\frac{1}{B} - \frac{1}{A}\right) x. \tag{15}$$

It is then an easy matter to find the solution to equations (13), which is

$$t = \frac{F(A) - G(B)}{A - B} \qquad x = \frac{BF(A) - AG(B)}{A - B} \tag{16}$$

where F and G are arbitrary functions of their arguments.

Moreover equations (12) can be put in a conservation-law form as

$$\partial_t T(A, B) + \partial_x X(A, B) = 0 \tag{17}$$

where

$$T = f'(A) - g'(B) \qquad X = Af'(A) - Bg'(B) - f(A) + g(B) \tag{18}$$

showing that there are infinite conservation laws which correspond to system (12). The prime in (18) denotes derivatives.

It is interesting, at this point, to make a comparison with the so-called 'linearly degenerate' systems described first by Lax (1957) and more recently by Boillat and Ruggeri (1978), in which

$$\frac{\partial A}{\partial U} R^A = \frac{\partial B}{\partial U} R^B = 0. \tag{19}$$

This fact implies that

$$\varphi^A L^A M = \frac{\partial B}{\partial U} \qquad \text{and} \qquad \varphi^B L^B M = \frac{\partial A}{\partial U} \tag{20}$$

where φ^A and φ^B are suitable integrating factors. So, the relevant Riemann invariants α and β are of the form

$$\alpha = \alpha(B) \qquad \beta = \beta(A). \tag{21}$$

The insertion of (20) into (3) yields, by the chain rule,

$$A_T + BA_X = 0 \qquad \text{and} \qquad B_T + AB_X = 0. \tag{22}$$

For reasons that will be apparent later on we have renamed the time and the spatial coordinate with the capital letters T and X.

When the hodograph transformation is performed one obtains the linear system (Ruggeri 1979)

$$X_A = AT_A \qquad \text{and} \qquad X_B = BT_B \tag{23}$$

the compatibility condition of which yields

$$X_{AB} = 0 \qquad (\text{or } T_{AB} = 0). \tag{24}$$

We may conclude that the Lax 'linearly degenerate' system (22) and those characterised by condition (8) share the property of being reducible to a hodograph equation integrable in an elementary way. Furthermore the two classes of 2×2 systems each have a Riemann invariant function of a single eigenvalue of the problem but interchanged (see (11) and (21)).

Proceeding with the comparison one finds that system (23) has the following solution (Ruggeri 1979):

$$T=f'(A)-g'(B) \qquad X=Af'(A)-Bg'(B)-f(A)+g(B) \tag{25}$$

(where the prime indicates derivatives) while system (22) admits an infinite number of conservation laws (Boillat and Ruggeri 1978)

$$\partial_T t(A, B)+\partial_X x(A, B)=0 \tag{26}$$

where the functions t and x have the form

$$t=\frac{F(A)-G(B)}{A-B} \qquad x=\frac{BF(A)-AG(B)}{A-B}. \tag{27}$$

We notice that, on one hand, (27) is nothing more than the solution of the hodograph system associated with equations (12) and, on the other hand, equations (25) have the same form as equation (18) which express the conservation laws of system (12).

The prototypes of the systems considered here come from classical gas dynamics. In particular, the flow characterised by a pressure law of the type $p=a+b^2\rho^3$ (p pressure, ρ mass density, a and b constants) satisfies condition (8). It is well known that in this case the equations of the isentropic one-dimensional flow split into (12) (cf Rozdestvenskii and Janenko 1981) and admit the general solution (16) (Gaffet 1983).

Another example, from the relativistic fluid dynamics, will be extensively treated elsewhere (Carbonaro 1989).

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